

A generalized structure of Bell inequalities for bipartite arbitrary-dimensional systems

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We propose a generalized structure of Bell inequalities for arbitrary d -dimensional bipartite systems, which includes the existing two types of Bell inequalities introduced by Collins-Gisin-Linden-Massar-Popescu [Phys. Rev. Lett. **88**, 040404 (2002)] and Son-Lee-Kim [Phys. Rev. Lett. **96**, 060406 (2006)]. We analyze Bell inequalities in terms of correlation functions and joint probabilities, and show that the coefficients of correlation functions and those of joint probabilities are in Fourier transform relations. We finally show that the coefficients in the generalized structure determine the characteristics of quantum violation and tightness.

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I. INTRODUCTION

Local-realistic theories impose constraints on any correlations obtained from measurement between two separated systems [1, 2, 3]. It was shown that these constraints, known as Bell inequalities, are incompatible with the quantitative predictions by quantum mechanics in case of entangled states. For example, the original Bell inequality is violated by a singlet state of two spin-1/2 particles [1]. The Clauser-Horne-Shimony-Holt (CHSH) inequality is another common form of Bell inequality, allowing more flexibility in local measurement configurations [2]. These constraints are of great importance for understanding the conceptual features of quantum mechanics and draw the boundary between local-realistic and quantum correlations. One may doubt if there is any well-defined constraint for many high-dimensional subsystems which would eventually simulate a classical system as increasing its dimensionality to infinity [3]. Therefore, constraints for more complex systems such as multi-partite or high-dimensional systems have been proposed and investigated intensively [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

For bipartite high-dimensional systems, Collins *et al.* suggested a local-realistic constraint, called CGLMP inequality [5]. It is violated by quantum mechanics and its characteristics of violation are consistent with the numerical results provided by Kaszlikowski *et al.* [6]. Further, Masanes showed that the CGLMP inequality is tight [7], which implies that the inequality has no interior bias as a local-realistic constraint. However, Acin *et al.* found that the CGLMP inequality shows maximal violation by non-maximally entangled state [8]. Zohren and Gill found the similar results when they applied CGLMP inequality to infinite dimensional systems [9]. Recently, Son *et al.* [10] suggested a generic Bell inequality and its variant for arbitrary high-dimensional systems. The variant will be called SLK inequality throughout this paper. They showed that the SLK inequality is maximally violated by maximally entangled state. Very recently, the CGLMP inequality was recasted in the structure of the SLK in-

equality by choosing appropriate coefficients [11].

In this paper, we propose a generalized structure of Bell inequalities for bipartite arbitrary d -dimensional systems, which includes various types of Bell inequalities proposed previously. A Bell inequality in the given generalized structure can be represented either in the correlation function space or joint probability space. We show that a Bell inequality in one space can be mapped into the other space by Fourier transformation. The two types of high-dimensional Bell inequalities, CGLMP and SLK, are represented in terms of the generalized structure with appropriate coefficients in both spaces (Sec. II). We investigate the violation of Bell inequalities by quantum mechanics. The expectations of local-realistic theories and quantum mechanics are determined by the coefficients of correlation functions or joint probabilities. The CGLMP inequality is maximally violated by non-maximally entangled state while the SLK is by maximally entangled state (Sec. III). We also investigate the tightness of Bell inequalities which represents whether they contain an interior bias or not at the boundary between local-realistic and quantum correlations. Then we show that the SLK is a non-tight Bell inequality while the CGLMP is tight (Sec. IV).

II. GENERALIZED ARBITRARY DIMENSIONAL BELL INEQUALITY

We generalize a Bell inequality for bipartite arbitrary d -dimensional systems. Suppose that each observer independently choose one of two observables denoted by A_1 or A_2 for Alice, and B_1 or B_2 for Bob. Here we associate a hermitian observables H to a unitary operator U by the simple correspondence, $U = \exp(iH)$, and call U a unitary observable [10, 12, 13, 14]. We note that unitary observable representation induces mathematical simplifications without altering physical results [19]. Each outcome takes the value of an element in the set of order d , $V = \{1, \omega, \dots, \omega^{d-1}\}$, where $\omega = \exp(2\pi i/d)$. The assumption of local-realistic theories implies that the out-

comes of observables are predetermined before measurements and the role of the measurements is just to reveal the values. The values are determined only by local hidden variables λ , i.e., $A_a(\lambda)$ and $B_b(\lambda)$ for $a, b = 1, 2$.

We denote a correlation between specific measurements taken by two observers, as $A_a(\lambda)B_b^*(\lambda)$. Based on the local hidden-variable description, the correlation function is the average over many trials of the experiment as

$$C_{ab} = \int d\lambda \rho(\lambda) A_a(\lambda) B_b^*(\lambda), \quad (1)$$

where $\rho(\lambda)$ is the statistical distribution of the hidden variables λ with the properties of $\rho(\lambda) \geq 0$ and $\int d\lambda \rho(\lambda) = 1$. The correlation function can be expanded in terms of joint probability functions over all possible outcome pairs (k, l) with complex-valued weight as

$$C_{ab} = \sum_{k,l=0}^{d-1} \omega^{k-l} P(A_a = k, B_b = l), \quad (2)$$

where ω^{k-l} is called a correlation weight and $P(A_a = k, B_b = l)$ is a joint probability of Alice and Bob obtaining outcomes ω^k and ω^l respectively. Here we use the powers k and l of the outcomes ω^k and ω^l for the arguments of the joint probability as there is one-to-one correspondence.

We assume in general a correlation weight $\mu_{k,l}$ to satisfy certain conditions [15]. [C.1] The correlation expectation vanishes for a bipartite system with a locally unpolarized subsystem: $\sum_k \mu_{k,l} = 0, \forall l$ and $\sum_l \mu_{k,l} = 0, \forall k$ [C.2] The correlation weight is unbiased over possible outcomes of each subsystem (translational symmetry within modulo d): $\mu_{k,l} = \mu_{k+\gamma, l+\gamma}, \forall \gamma$. [C.3] The correlation weight is uniformly distributed modulo d : $|\mu_{k+1,l} - \mu_{k,l}| = |\mu_{k,l+1} - \mu_{k,l}|, \forall k, l$. The correlation weight in Eq. (2) ω^{k-l} satisfies all the conditions, can be written as ω^α where $\alpha \equiv k - l \in \{0, 1, \dots, d-1\}$ and it obeys $\sum_\alpha \omega^\alpha = 0$.

Let us now consider higher-order(n) correlations following also the local hidden-variable description. The n -th order correlation function averaged over many trials of the experiment corresponds to the n -th power of 1-st order correlation as

$$\begin{aligned} C_{ab}^{(n)} &= \int d\lambda \rho(\lambda) (A_a(\lambda) B_b^*(\lambda))^n \\ &= \sum_{k,l=0}^{d-1} \omega^{n(k-l)} P(A_a = k, B_b = l) \\ &= \sum_{\alpha=0}^{d-1} \omega^{n\alpha} P(A_a \doteq B_b + \alpha), \end{aligned} \quad (3)$$

where the n -th order correlation weight $\omega^{n\alpha}$ also satisfies the above conditions, C.1, C.2 and C.3, and $P(A_a \doteq B_b + \alpha)$ is the joint probability of local measurement outcomes differing by a positive residue α modulo d . Here we

note that the higher-order correlations Eq. (3) show the periodicity of $C_{ab}^{(d+n)} = C_{ab}^{(n)}$ and they have the Fourier relation with the joint probabilities as given in Eq. (3).

We present a generalized Bell function for arbitrary d -dimensional system using higher-order correlation functions as

$$\mathcal{B} = \sum_{a,b} \sum_{n=0}^{d-1} f_{ab}(n) C_{ab}^{(n)}, \quad (4)$$

where coefficients $f_{ab}(n)$ are functions of the correlation order n and the measurement configurations a, b . They determine the constraint of local-realistic theories with a certain upper bound and its violation by quantum mechanics will be investigated in Sec. III. The zero-th order correlation has no meaning as it simply shift the value of \mathcal{B} by a constant and is chosen to vanish, i.e., $\sum_{a,b} f_{ab}(0) = 0$. The Bell function in Eq. (4) is rewritten in terms of the joint probabilities given in Eq. (3), as

$$\mathcal{B} = \sum_{a,b} \sum_{\alpha=0}^{d-1} \epsilon_{ab}(\alpha) P(A_a \doteq B_b + \alpha), \quad (5)$$

where $\epsilon_{ab}(\alpha)$ are coefficients of the joint probabilities $P(A_a \doteq B_b + \alpha)$.

We note that the coefficients $\epsilon_{ab}(\alpha)$ are obtained by the Fourier transformation of $f_{ab}(n)$ based on the kernel of a given correlation weight as

$$\epsilon_{ab}(\alpha) = \sum_{n=0}^{d-1} f_{ab}(n) \omega^{n\alpha}, \quad (6)$$

$$f_{ab}(n) = \frac{1}{d} \sum_{\alpha=0}^{d-1} \epsilon_{ab}(\alpha) \omega^{-n\alpha}. \quad (7)$$

It is remarkable that one can represent a given Bell function either in the correlation function space or joint probability space by using the Fourier transformation of the coefficients between them. This is the generalization of the Fourier transformation in 2-dimensional Bell inequalities provided by Werner *et al.* [16].

Different Bell inequalities can be represented by altering coefficients of the generalized structure, including previously proposed Bell inequalities in bipartite systems. In the case of $d = 2$, CHSH-type inequalities can be obtained with coefficients as $f(1) = (1, 1, -1, 1)$ and $\epsilon_{ab}(\alpha) = f_{ab}(1)(-1)^\alpha$ where $\alpha \in \{0, 1\}$. For arbitrary d -dimensional systems, the two types of Bell inequalities, CGLMP and SLK, are represented in terms of the generalized structure with appropriate coefficients obtained as follows.

CGLMP inequality - As it was originally proposed in terms of joint probabilities [5], the Bell function of the CGLMP inequality is in the form of (5) and its coeffi-

cients are given as

$$\begin{aligned}\epsilon_{11}(\alpha) &= 1 - \frac{2\alpha}{d-1}, & \epsilon_{12}(\alpha) &= -1 + \frac{2(\alpha-1)}{d-1}, \\ \epsilon_{21}(\alpha) &= -1 + \frac{2\alpha}{d-1}, & \epsilon_{22}(\alpha) &= 1 - \frac{2\alpha}{d-1},\end{aligned}\quad (8)$$

where the dot implies the positive residue modulo d . By using the inverse Fourier transformation in Eq. (7) the coefficients for the correlation function representation are obtained as

$$\begin{aligned}f_{11}(n \neq 0) &= \frac{2}{d-1} \left(\frac{1}{1-\omega^{-n}} \right), \\ f_{12}(n \neq 0) &= \frac{2}{d-1} \left(\frac{1}{1-\omega^n} \right), \\ f_{21}(n \neq 0) &= \frac{2}{d-1} \left(\frac{1}{\omega^{-n}-1} \right), \\ f_{22}(n \neq 0) &= \frac{2}{d-1} \left(\frac{1}{1-\omega^{-n}} \right), \\ f_{ab}(n=0) &= 0 \quad \forall a, b,\end{aligned}\quad (9)$$

where the sum of the 0-th order coefficients vanishes and does not affect the characteristics of the Bell inequality.

SLK inequality - It was introduced in terms of correlation functions [10], and the coefficients are given by

$$\begin{aligned}f_{11}(n \neq 0) &= (\omega^{n\delta} + \omega^{(n-d)\delta})/4, \\ f_{12}(n \neq 0) &= (\omega^{n(\delta+\eta_1)} + \omega^{(n-d)(\delta+\eta_1)})/4, \\ f_{21}(n \neq 0) &= (\omega^{n(\delta+\eta_2)} + \omega^{(n-d)(\delta+\eta_2)})/4, \\ f_{22}(n \neq 0) &= (\omega^{n(\delta+\eta_1+\eta_2)} + \omega^{(n-d)(\delta+\eta_1+\eta_2)})/4, \\ f_{ab}(n=0) &= 0 \quad \forall a, b,\end{aligned}\quad (10)$$

where δ is a real number, called a variant factor, and $\eta_{1,2} \in \{+1/2, -1/2\}$. By varying δ and $\eta_{1,2}$, one can have many variants of SLK inequalities. For all the variants the coefficients in the joint probability picture are obtained as

$$\begin{aligned}\epsilon_{11}(\alpha) &= S(\delta + \alpha), \\ \epsilon_{12}(\alpha) &= S(\delta + \eta_1 + \alpha), \\ \epsilon_{21}(\alpha) &= S(\delta + \eta_2 + \alpha), \\ \epsilon_{22}(\alpha) &= S(\delta + \eta_1 + \eta_2 + \alpha),\end{aligned}\quad (11)$$

where

$$\begin{aligned}S(x \neq 0) &= \frac{1}{4}(\cot \frac{\pi}{d} x \sin 2\pi x - \cos 2\pi x - 1), \\ S(x=0) &= \frac{1}{2}(d-1).\end{aligned}\quad (12)$$

We have shown that those two types of high-dimensional inequalities have different coefficients but the same generalized structure. In the frame work of the generalized structure we will now study how the coefficients determine the characteristics of Bell inequalities such as the degree of violation and tightness.

III. VIOLATION BY QUANTUM MECHANICS

In order to see the violation of Bell inequalities by quantum mechanics we need to know the upper bound by local hidden variable theories. We note that a probabilistic expectation of a Bell function is given by the convex combination of all possible deterministic values of the Bell function and the local-realistic upper bound is decided by the maximal deterministic value. Let $\alpha_{ab} = \alpha$ such that $P(A_a \doteq B_b + \alpha) = 1$. The assumption of local-realistic theories implies that the values α_{ab} are predetermined. For a Bell function in the form of Eq. (4) or (5), they obey the constraint, $\alpha_{11} + \alpha_{22} \doteq \alpha_{12} + \alpha_{21}$, because of the identity, $A_1 - B_1 + A_2 - B_2 = A_1 - B_2 + A_2 - B_1$. The local-realistic upper bound of the Bell function is therefore given by

$$\mathcal{B}_{\text{LR}}^{\text{max}} = \max_{\alpha_{ab}} \left[\sum_{a,b} \epsilon_{ab}(\alpha_{ab}) |\alpha_{11} + \alpha_{22} \doteq \alpha_{12} + \alpha_{21}| \right]. \quad (13)$$

The quantum expectation value for arbitrary quantum state $\hat{\rho}$ is written by

$$\begin{aligned}\mathcal{B}_{\text{QM}}(\hat{\rho}) &= \text{Tr}(\hat{\mathcal{B}}\hat{\rho}) \\ &= \sum_{a,b} \sum_{n=0}^{d-1} f_{ab}(n) \text{Tr}(\hat{C}_{ab}^{(n)} \hat{\rho}),\end{aligned}\quad (14)$$

where $\hat{\mathcal{B}}$ is the Bell operator defined by replacing the correlation function in Eq. (4) with correlation operator, $\hat{C}_{ab}^{(n)} = \sum_{k,l} \omega^{n(k-l)} \hat{P}_a \otimes \hat{P}_b$ where \hat{P}_a, \hat{P}_b are projectors onto the measurement basis denoted by a, b . If an expectation value of any quantum state exceeds the local realistic bound $\mathcal{B}_{\text{LR}}^{\text{max}}$, i.e., the Bell inequality is violated by quantum mechanics, the composite system is entangled and shows nonlocal quantum correlations. The maximal quantum expectation is called quantum maximum $\mathcal{B}_{\text{QM}}^{\text{max}}$ and corresponds to the maximal eigenvalue of the Bell operator. In the case of $d=2$, with the coefficients $f(1) = (1, 1, -1, 1)$ we can obtain the quantum maximum, $\mathcal{B}_{\text{QM}}^{\text{max}} = 2\sqrt{2}$, which is in agreement with the Cirel'son bound [20].

In the presence of white noise, a maximally entangled state $|\psi_m\rangle$ becomes $\hat{\rho} = p|\psi_m\rangle\langle\psi_m| + (1-p)\mathbb{1}/d^2$ where p is the probability that the state is unaffected by noise. Then, the minimal probability for the violation is $p^{\text{min}} = \mathcal{B}_{\text{LR}}^{\text{max}}/\mathcal{B}_{\text{QM}}(|\psi_m\rangle)$. We now investigate the violation of two types of Bell inequalities, CGLMP and SLK, and compare them as follows.

CGLMP inequality - The local-realistic upper bound, $\mathcal{B}_{\text{LR}}^{\text{max}} = 2$, can be obtained as Eq. (13). The quantum expectation can also be obtained as Eq. (14) and it is consistent with the result in Ref. [5]. Acin *et al.* found, however, that the CGLMP inequality shows maximal violation for non-maximally entangled states [8]. For 3-dimensional system, the quantum maximum is $\mathcal{B}_{\text{QM}}^{\text{max}} \simeq 2.9149$ for the non-maximally entangled state,

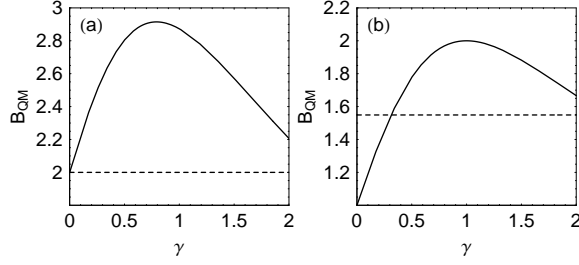


FIG. 1: The expectation value of (a) the CGLMP and (b) the optimal SLK for $d = 3$ as varying the value γ for the quantum state, $(1/\sqrt{n})(|00\rangle + \gamma|11\rangle + |22\rangle)$ where $n = 2 + \gamma^2$. The SLK takes the maximum 2 when the state is maximally entangled ($\gamma = 1$), whereas the CGLMP takes the maximum 2.9149 for a partially entangled state ($\gamma \simeq 0.7923$). The dashed lines indicate the local-realistic upper bounds.

$(1/\sqrt{n})(|00\rangle + \gamma|11\rangle + |22\rangle)$ where $\gamma \simeq 0.7923$ and $n = 2 + \gamma^2$. It is higher than the expectation by maximally entangled state, $\mathcal{B}(|\psi_m\rangle) \simeq 2.8729$. The expectation of the CGLMP is shown in Fig. 1 against the entanglement degree γ , once the local measurements are chosen such that they maximize the Bell function for the maximally entangled state. Further, we also note that the minimal violation probability (p^{\min}) of the CGLMP decreases as the dimension d increases.

SLK inequality - Many variants of the SLK Bell inequality are obtained by varying δ and $\eta_{1,2}$. All the variants of the SLK have the same quantum maximum $d - 1$ for a maximally entangled state $|\psi_m\rangle$, $\mathcal{B}_{\text{QM}}^{\max} = \mathcal{B}_{\text{QM}}(|\psi_m\rangle)$, as we prove in Appendix . On the other hand, the local-realistic upper bounds depend on the variants. The local-realistic upper bound $\mathcal{B}_{\text{LR}}^{\max}$ is a function of the variant factor δ . It shows a periodicity, $\mathcal{B}_{\text{LR}}^{\max}(\delta) = \mathcal{B}_{\text{LR}}^{\max}(\delta + 1/2)$, and without loss of generality it suffices to consider $0 \leq \delta < 1/2$. If $\delta = 0$, the local-realistic upper bound is the same as the quantum maximum $d - 1$, and thus the corresponding Bell inequality is not violated by quantum mechanics. When $\delta = 1/4$, we have the lowest local-realistic upper bound as

$$\min_{\delta} [\mathcal{B}_{\text{LR}}^{\max}(\delta)] = \frac{1}{4} \left(3 \cot \frac{\pi}{4d} - \cot \frac{3\pi}{4d} \right) - 1, \quad (15)$$

and for other cases the bound values are symmetric at $\delta = 1/4$, i.e., $\mathcal{B}_{\text{LR}}^{\max}(1/4 + \epsilon) = \mathcal{B}_{\text{LR}}^{\max}(1/4 - \epsilon)$ for $0 < \epsilon \leq 1/4$. Therefore, we will call the variant of $\delta = 1/4$, which gives the maximal difference between quantum maximum and local-realistic upper bound, as the *optimal* SLK inequality and use it for comparing to the CGLMP. In Fig. 1, we present the quantum expectation of the SLK for 3-dimensional systems against the degree γ , where the local measurements are chosen such that they maximize the Bell function for the maximally entangled state. Note that the SLK inequality shows the maximal violation by maximally entangled states and the minimal violation probability p^{\min} increases as the di-

mension d increases.

By investigating the violation of two inequalities, CGLMP and SLK, based on the generalized structure of Bell inequalities, we showed that those two types have very different characteristics. The SLK inequality is maximally violated by maximally entangled states as being consistent with our intuition whereas the CGLMP is maximally violated by non-maximally entangled states. We remark that the coefficients of the given generalized structure determine the characteristics of quantum violations.

IV. TIGHTNESS OF BELL INEQUALITIES

The set of possible outcomes for a given measurement setting forms a convex polytope in the joint probability space or alternatively in the correlation function space [7, 16, 17, 18]. Each generator of the polytope represents the predetermined measurement outcome called local-realistic configuration. All interior points of the polytope are given by the convex combination of generators and they represent the accessible region of local-realistic theories associated with the probabilistic expectations of measurement outcomes. Therefore, every facet of the polytope is a boundary of halfspace characterized by a linear inequality, which we call *tight Bell inequality*. There are non-tight Bell inequalities which contain the polytope in its halfspace. As the non-tight Bell inequality has interior bias at the boundary between local-realistic and quantum correlations, one might say it to be the worse detector of the nonlocal test [7, 16, 17].

The Bell polytope is lying in the joint probability space of dimension h , the degrees of freedom for the measurement raw data. For a bipartite system, two observables per party and d -dimensional outcomes, the joint probability, $P(A_a = k, B_b = l)$ where $k, l = 0, 1, \dots, d - 1$ and $a, b = 1, 2$, can be arranged in a $4d^2$ -dimensional vector space. However, the joint probabilities have two constraints, i.e., normalization and no-signaling constraints, which reduce dimension by $4d$ [7]. The generators in the h -dimensional space can be written, following the notations in Ref. [7], as

$$\mathbf{G} = |A_1, B_1\rangle \oplus |A_1, B_2\rangle \oplus |A_2, B_1\rangle \oplus |A_2, B_2\rangle \quad (16)$$

where $|n\rangle$ stands for $|n \bmod d\rangle$ and is the d -dimensional vector with a 1 in the n -th component and 0s in the rest.

In order to examine the tightness of a given generalized Bell inequality, in general one considers the following conditions that every tight Bell inequality fulfills [7]. [T.1] All the generators must belong to the half space of a given facet. [T.2] Among the generators on the facet, there must be h which are linearly independent. First, it is straightforward that all generators fulfill the inequality as the Bell inequality derived to do. As the local-realistic upper bound is the maximum among expectation values of local-realistic configurations, all generators are located below the local-realistic upper bound, $\mathcal{B}_{\text{LR}}^{\max}$.

Thus the first condition T.1 is fulfilled. Second, we examine whether there are h linearly independent generators which give the value of the local-realistic bound, $\mathcal{B}_{\text{LR}}^{\text{max}}$. By the predetermined local-realistic values α_{ab} , the generators (16) become

$$|A, A - \alpha_{11}\rangle \oplus |A, A - \alpha_{12}\rangle \oplus |A - \alpha_{12} + \alpha_{22}, A - \alpha_{11}\rangle \\ \oplus |A - \alpha_{11} + \alpha_{21}, A - \alpha_{12}\rangle, \quad (17)$$

where $A \in \{0, 1, \dots, d-1\}$ and the number of linearly independent generators is determined by the number of sets $\{\alpha_{ab}\}$ that give the local-realistic upper bound. If the number of linear independent generators is not smaller than $h = 4d(d-1)$, the corresponding Bell inequality is tight.

CGLMP inequality - For the CGLMP inequality the local-realistic upper bound is achieved when $\alpha_{11} + \alpha_{22} - (\alpha_{12} - 1) - \alpha_{21} + d - 1 = 0$. The condition allows the sufficient number of linearly independent generators and the CGLMP inequality is tight [7].

SLK inequality - For the optimal SLK inequality, the upper bound is obtained in the case that $\{\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}\}$ is equal to one of four sets; $\{0, 0, d-1, d-1\}$, $\{0, 0, 0, 0\}$, $\{0, 1, d-1, 0\}$, $\{d-1, 0, d-1, 0\}$. Thus there are four types of generators as

$$|A, A\rangle \oplus |A, A\rangle \oplus |A-1, A\rangle \oplus |A-1, A\rangle \\ |A, A\rangle \oplus |A, A\rangle \oplus |A, A\rangle \oplus |A, A\rangle \\ |A, A\rangle \oplus |A, A-1\rangle \oplus |A-1, A\rangle \oplus |A-1, A-1\rangle \\ |A, A+1\rangle \oplus |A, A\rangle \oplus |A, A+1\rangle \oplus |A, A\rangle \quad (18)$$

which are linearly independent with $A \in \{0, 1, \dots, d-1\}$. There are only $4d$ linearly independent generators which are smaller than $h = 4d(d-1)$, the tightness condition T.2. Thus the optimal SLK inequality is non-tight. On the other hand, the SLK inequality for $\delta = 0$ is tight but it is not violated by quantum mechanics.

V. REMARKS

In summary, we presented a generalized structure of the Bell inequalities for arbitrary d -dimensional bipartite systems by considering the correlation function specified by a well-defined complex-valued correlation weight. The coefficients of a given Bell inequality in the correlation function space and the joint probability space were shown to be in the Fourier relation. Two known types of high-dimensional Bell inequalities, CGLMP and SLK, were shown to have the generalized structure in common and we found their coefficients in both spaces.

Based on the generalized structure, we investigated characteristics of the Bell inequalities such as quantum violation and tightness. We found that the CGLMP and SLK inequalities show different characteristics. For instance, the SLK inequality is maximally violated by maximally entangled states, which is consistent with the intuition “the larger entanglement, the stronger violation

against local-realistic theories,” whereas the CGLMP inequality is maximally violated by the non-maximally entangled state as previously shown by Acin *et al.* [8]. On the other hand, in analyzing the tightness of the inequalities, the CGLMP is tight but the SLK inequality is found to be non-tight for $\delta \neq 0$, implying that the SLK inequality has interior bias at the boundary between local-realistic and quantum correlations.

The correlation coefficients of Bell inequalities play a crucial role in determining their characteristics of quantum violation and tightness. This implies that by altering the coefficients in the generalized structure one can construct other Bell inequalities. The present work opens a possibility of finding a new Bell inequality that fulfills both conditions of the maximal violation by maximal entanglement and the tightness.

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APPENDIX: QUANTUM MAXIMUM OF ALL VARIANT SLK INEQUALITIES

We shall prove that all the variant SLK inequalities take $d-1$ as the quantum maximum. The Bell operator of the SLK variants can be written as

$$\hat{\mathcal{B}}_S = \frac{1}{2} \sum_{n=1}^{d-1} \alpha \cdot \tilde{\beta}, \quad (\text{A.1})$$

where $\alpha = (\hat{A}_1^{\dagger n}, \hat{A}_2^{\dagger n})^T$ and $\tilde{\beta} = \mathbf{U}\beta$ with $\beta = (\hat{B}_1^n, \hat{B}_2^n)^T$ and \mathbf{U} is a 2×2 unitary matrix with elements,

$$U_{11} = (\omega^{n\delta} + \omega^{(n-d)\delta})/2, \\ U_{12} = (\omega^{n(\delta+\eta_1)} + \omega^{(n-d)(\delta+\eta_1)})/2, \\ U_{21} = (\omega^{n(\delta+\eta_2)} + \omega^{(n-d)(\delta+\eta_2)})/2, \\ U_{22} = (\omega^{n(\delta+\eta_1+\eta_2)} + \omega^{(n-d)(\delta+\eta_1+\eta_2)})/2, \quad (\text{A.2})$$

where $\eta_{1,2} \in \{1/2, -1/2\}$.

The expectation of the Bell operator is given by

$$\begin{aligned}
& \frac{1}{2} \left| \sum_{n=1}^{d-1} \langle \psi | \alpha \cdot \tilde{\beta} | \psi \rangle \right| \leq \frac{1}{2} \sum_{n=1}^{d-1} \left| \langle \psi | \alpha \cdot \tilde{\beta} | \psi \rangle \right| \\
& \leq \frac{1}{2} \sum_{n=1}^{d-1} \left(\left| \langle \psi | \alpha_1 \otimes \tilde{\beta}_1 | \psi \rangle \right| + \left| \langle \psi | \alpha_2 \otimes \tilde{\beta}_2 | \psi \rangle \right| \right) \\
& \leq \frac{1}{\sqrt{2}} \sum_{n=1}^{d-1} \sqrt{\left| \langle \psi | \alpha_1 \otimes \tilde{\beta}_1 | \psi \rangle \right|^2 + \left| \langle \psi | \alpha_2 \otimes \tilde{\beta}_2 | \psi \rangle \right|^2} \\
& = \frac{1}{\sqrt{2}} \sum_{n=1}^{d-1} \sqrt{\sum_{i=1}^2 \left| \langle \psi | \alpha_i \otimes \tilde{\beta}_i | \psi \rangle \right|^2}, \tag{A.3}
\end{aligned}$$

where we consecutively used the triangle inequality and the arithmetic-geometric means inequality, $2|a||b| \leq |a|^2 + |b|^2$. Note that

$$\begin{aligned}
\sum_{i=1}^2 \left| \langle \psi | \alpha_i \otimes \tilde{\beta}_i | \psi \rangle \right|^2 & \leq \sum_{i=1}^2 \langle \psi | \left(\alpha_i^\dagger \otimes \tilde{\beta}_i^\dagger \right) \left(\alpha_i \otimes \tilde{\beta}_i \right) | \psi \rangle \\
& = \sum_{i=1}^2 \langle \psi | \mathbb{1} \otimes \tilde{\beta}_i^\dagger \tilde{\beta}_i | \psi \rangle, \tag{A.4}
\end{aligned}$$

where we used that α_i is unitary. Here the above inequality is obtained by reasoning that $\hat{Q} \equiv \mathbb{1} - |\psi\rangle\langle\psi|$ is a positive operator as $\langle\phi|\hat{Q}|\phi\rangle = 1 - |\langle\phi|\psi\rangle|^2 \geq 0$ for any $|\phi\rangle$, and $|\langle\psi|\hat{C}|\psi\rangle|^2 = \langle\psi|\hat{C}^\dagger|\psi\rangle\langle\psi|\hat{C}|\psi\rangle = \langle\psi|\hat{C}^\dagger(\mathbb{1} - \hat{Q})\hat{C}|\psi\rangle = \langle\psi|\hat{C}^\dagger\hat{C}|\psi\rangle - \langle\psi|\hat{Q}\hat{C}|\psi\rangle \leq \langle\psi|\hat{C}^\dagger\hat{C}|\psi\rangle$, where $\hat{C} \equiv \alpha_i \otimes \tilde{\beta}_i$. Since $\sum_i \tilde{\beta}_i^\dagger \tilde{\beta}_i = \sum_{jk} \sum_i U_{ij}^* U_{ik} \beta_j^\dagger \beta_k = \sum_{jk} \delta_{jk} \beta_j^\dagger \beta_k = \sum_i \beta_i^\dagger \beta_i = 2\mathbb{1}$, it is clear that

$$\sum_{i=1}^2 \left| \langle \psi | \alpha_i \otimes \tilde{\beta}_i | \psi \rangle \right|^2 \leq \langle \psi | \mathbb{1} \otimes \sum_{i=1}^2 \beta_i^\dagger \beta_i | \psi \rangle = 2. \tag{A.5}$$

Hence the upper bound for all variants of the SLK is

$$\left| \langle \psi | \hat{\mathcal{B}}_S | \psi \rangle \right| \leq d - 1. \tag{A.6}$$

Since all SLK Bell operators have the eigenvalue $d - 1$ for maximally entangled states, the upper bound is reachable. Therefore, $d - 1$ is the quantum maximum for all variants of the SLK inequality.

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